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Manuscript received February 6, 1979; revision received August 26, and accepted August 27, 1979.

Part II: Structural Aspects and the Synthesis of Alternative Feasible Control Schemes

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The classification of control objectives and external disturbances in a chemical plant determines the extent of the optimizing and regulatory control structures (see Part I). In this article we discuss the structural design of alternative regulatory control schemes to satisfy the posed objective. Within the framework of hierarchical control, criteria are developed for the further decomposition of the process subsystems, reducing the combinatorial problem while not eliminating feasible control structures. We use structural models to describe the interactions among the units of a plant and the physicochemical phenomena occurring in the various units. The relevance of controllability and observability in the synthesis of control structures is discussed, and modified versions are used to develop all the alternative feasible regulatory structures in an algorithmic fashion. Various examples illustrate the developed concepts and strategies, including the application of the overall synthesis method to an integrated chemical plant.

SCOPE

In Part I, the control tasks were divided into those of the regulatory and of the optimizing type. The first can always be expressed in the form of functions of operating variables, which have to be kept at desired levels through the use manipulated variables. The same is possible for the second, if certain conditions derived in Part I are satisfied. The structure of the feedback controllers used for that purpose is developed here on a sound theoretical basis. The following problems are addressed:

1. Development of a suitable type of system representation (model), requiring a minimal amount of information.

2. Formulation of mathematical criteria to be satisfied by every feasible control structure.

3. Development of guidelines for decomposing the overall problem into manageable subproblems.

4. Algorithmic procedure to develop alternative control structures.

The approach adopted in this work is based on the structural characteristics describing (a) the interactions among the units of a chemical process and (b) the logical dependence (of the Boolean type) among the variables used to model the dynamic behavior of the various units. Thus, detailed dynamic modelling at an early stage is avoided. The mathematical feasibility criteria for the generated alternative control structures are

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based on the concepts of controllability and observability. But they go far beyond the general textbook definitions, which are demonstrated to be too weak in certain cases and too strong in others.

The process decomposition developed in Part I for the synthesis of decentralized optimizing control structures can be extended for the design of the regulatory structures. Within the above framework, we develop an algorithmic procedure to generate alternative control structures for a given steady-state chemical process design. We demonstrate the synthesis strategies in a series of examples.

CONCLUSIONS AND SIGNIFICANCE

A method for synthesizing control structures is presented. Systems are represented as structured matrices and the method uses extended versions of the conditions for structural controllability and observability as feasibility criteria. Mainly, feedback control structures are addressed, and feedforward compensation is developed as a logical extension. Numerous examples show the various characteristics of the synthesis method. A hypothetical plant (Williams and Otto 1960) demonstrates the applicability of the method.

The significance of the paper is twofold: First, it presents a theory-based method for developing alternative control structures—excluding the possibility of singularities, overspecifications and undetectable local instabilities. Engineering heuristic criteria can enter at any stage of the synthesis procedure, and thus, the method should be suitable for industrial applications. Second, the usefulness and implication of the properties, controllability and observability, have been only incompletely covered and understood in the chemical engineering literature. Here a unified representation is given, explaining the physical meaning of those concepts.

The results are by no means final, but they are a first step to bridge the discontinuity between the employed completely heuristic structuring procedures and the available sophisticated detailed design techniques for multivariable control loops.

In Part I we presented the general philosophy for the synthesis of control structures for chemical processes. The steps are summarized in Figure 5 of Part I. Specification of the control objectives is the first step, and dictates that the control task be composed of a regulatory part and an optimizing part. Classification of the expected disturbances determines the extent of each of these control parts.

Part I concentrated further on optimizing controllers after developing the proper decomposition of the process. The implicit assumption made at that point is that the regulatory controllers would be designed for each subsystem developed from the decomposition, but no guidelines were established on how to develop these regulatory structures. It is this question that we mainly address here.

Regulatory control structures for a chemical process are designed to keep certain processing variables at desired set points. These set points might be the results of product quality specifications, safety considerations, environmental regulations etc. or from feedback optimizing structures. These objectives should be satisfied almost continuously, and indicate the set of measurements that should be made in the process. In case these measurements cannot be made for various reasons—because of undesirable large time lags or low reliability, we select secondary measurements in conjunction with an estimation scheme (Part III).

As a first step we have to determine what variables have to be measured and controlled to guarantee smooth plant operation. Before the actual control algorithms can be designed, the alternative sets of manipulated variables which can be used in a feedback arrangement must be developed.

Whenever we approach a design problem, establishing suitable process models offers great difficulties. The underlying philosophy is to make initial decisions based on a crude model, and refine the model appropriately after each design step. The use of a simple model at the start is adopted here. The most primitive model for control purposes is one which displays the structural dependencies of the variables only, showing if the time derivative of one variable depends on another or not. For initial structural considerations, this turns out to be sufficient.

We found that the most efficient way to determine feasible sets of measured and manipulated variables, and to keep the model as simple as possible, is to use criteria of structural controllability and observability (Lin 1976). These properties, however, are neither necessary nor sufficient for a control system to work in practice. Extended concepts of output structural controllability and observability have been formulated to remedy these deficiencies

MODELING ASPECTS

Most results of the current systems theory are based on linear or linearized system models

$$\dot{x} = A(t)x + B(t)u \quad (1)$$

$$y = C(t)x \quad (2)$$

($x \in R^n$, $u \in R^m$, $y \in R^r$ and $A(t)$, $B(t)$, $C(t)$ are compatible matrices). Consequently, the linearized description of a nonlinear system will be accurate in an infinitesimal region around the point of linearization only. A compromise would be to use a linear model which is approximately valid in a finite region of the state space. Most of the elements in the matrices A, B and C will vary from one linearization point to another, but some of the elements will always be zero. This suggests a structural representation of the system, where A, B and C consist of elements which are generally nonzero and others which are always zero. The model of a double effect evaporator (DEE) (Figure 1) is

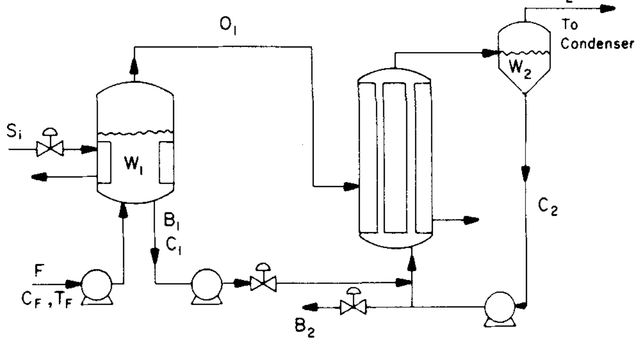


Figure 1. The double effect evaporator.

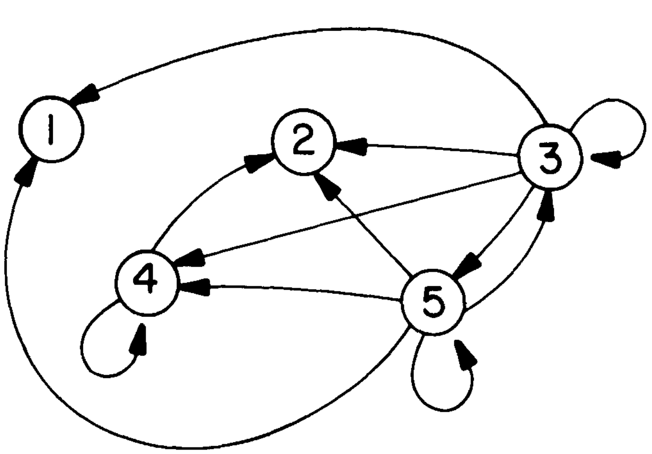


Figure 2. The digraph for the double effect evaporator.

used as an illustrative example at this point. Developing the structural linearized equations is quite straightforward, and is given by Newell and Fisher (1972).

$\dot{W}_1 = f_1(C_1,h_1;F,B_1)$
$\dot{W}_2 = f_2(C_1,C_2,h_1;B_1,B_2)$
$\dot{C}_1 = f_3(C_1,h_1;F,C_F)$
$\dot{C}_2 = f_4(C_1,C_2,h_1;B_1)$
$\dot{h}_1 = f_5(C_1,h_1;F,T_F,S)$

where:

W_1 holdup in the first effect
W_2 holdup in the second effect
C_1 concentration in the first effect
C_2 concentration in the second effect
h_1 enthalpy in the first effect
F feedflowrate
C_F concentration in feedstream
T_F feed temperature
S_1 steam rate
B_1 outlet flowrate of first effect
B_2 outlet flowrate of second effect

The structural matrix of this system is

	W_1	W_2	C_1	C_2	h_1	F	C_F	T_F	S_i	B_1	B_2
$\dot{W}_1$			X		X	X				X	
$\dot{W}_2$			X	X	X					X	X
$\dot{C}_1$			X		X	X	X				
$\dot{C}_2$			X	X	X					X	
$\dot{h}_1$			X		X	X		X	X		

The first part corresponds to the system matrix A, the controller matrix B will consist of a subset of the columns of the second part (e.g., B_1, B_2, F) while the remaining columns indicate the influence of disturbances. We can associate a graph with the system matrix which shows the mutual influence of the variables. The system variables form the state-nodes of the graph. There is a directed edge from node j to node i, if the structural system matrix has a nonzero entry in the i^{th} row and the j^{th} column. The graph corresponding to the structural matrix above is shown in Figure 2. Each manipulated variable and disturbance can be represented by a node, and its influence on the state variables shown graphically. The structural representation of a staged system gives rise to large matrices with repeated common structural elements. This observation has been used to reduce the size of the representation of complex staged systems, without losing the fundamental information about the system structure (see Appendix C).

In connection with structural matrices and their associated graphs, we define the generic rank ρ_g of a structural matrix to be the maximal rank a matrix achieves as a function of its free parameters; e.g., the matrix

$$\begin{pmatrix} X & 0 \\ X & X \end{pmatrix}$$

has a generic rank of 2 despite the fact that one of the diagonal elements could be zero as a special case and then the rank would be 1.

We define a node i to be nonacessible from a node j if there is no possibility of reaching node i starting from node j and going to node i only in the direction of the arrows along a path in the graph.

Govind and Powers (1976) used a steady-state cause and effect graph as their basic modeling for the synthesis of control structures and accessibility, as one of the feasibility criteria that a control structure should satisfy. We show this later to be insufficient, i.e., additional modeling is needed and further feasibility criteria should be imposed.

FEASIBILITY ANALYSIS OF CONTROL STRUCTURES

Apart from the available engineering rules of thumb, we would like to establish rigorous mathematical critieria to decide the feasibility of a suggested control structure. Every regulatory control structure should achieve the following two objectives: (i) Be it a regulation or a servo problem, the goal is to bring several outputs of the system from an undesired state to the desired one in some tolerable fashion, under the influence of disturbances entering the system. (ii) On the other hand, we would like to monitor the process almost continuously by observing all those outputs which are critical for system performance.

The concepts of complete state controllability and observability are generally believed to provide the necessary and sufficient conditions for a control structure to achieve the above two objectives. This is not so, and in the following paragraphs, we discuss the shortcomings of these concepts.

CONTROLLABILITY AND OBSERVABILITY IN RELATION TO PROCESS CONTROL STRUCTURES

The concept of controllability was introduced by Kalman (1960) and states that a linear system with the state differential equation

$$\dot{x}(t) = A(t) \times (t) + B(t)\, u(t) \quad (1)$$

is said to be completely controllable if the state of the system can be transferred from the zero state at any initial time, to any terminal state $x(t_1) = x_1$ within a finite time $t_1 - t_0$, through the use of a piecewise continuous control input $u(t)$.

If A and B are constant matrices, it is well known that the pair (A,B) is completely controllable if and only if

$$\text{rank}\ (B, AB, A^2B, \ldots, A^{n-1}B) = n \quad (3)$$

The mathematical implications of controllability with respect to existence and uniqueness of an optimal feedback control, eigenvalue assignment using state variable feedback etc. are available in any textbook (Kwakernaak and Sivan 1972). We would like to explore the usefulness of controllability for the synthesis of control structures. To avoid common misunderstandings, let us clarify the deficiencies of the definition and theorem above for our purposes:

1. The path for going from x_0 to x_1 is not completely arbitrary. Any practical control input might result in intolerably large deviations before we reach x_1 from x_0.
2. If the control u is bounded, we might not be able to reach x_1 in the specified amount of time, or not at all.
3. We have no information concerning regulation, i.e., what to do when disturbances enter the system.
4. Assuming that we are just interested in keeping a subset of the outputs at their setpoints in the face of disturbances by constructing feedback loops using the available manipulated variables, we cannot deduce if and how this can be achieved.
5. The rank test gives us no quantitative clues as to "how controllable" a system is.
6. The rank test might fail because of some unfortunate parameter choice. In reality, most of the system parameters are determined experimentally and not known exactly. An arbitrarily small variation in some of the parameters might make the system controllable.

These remarks might create the impression that to require a control structure to satisfy controllability is insufficient, but there are examples that controllability is more than what we should ask for (Rosenbrock 1976). Let a plant consist of two stirred tanks in series. The pH in the outlet of the second tank is to be adjusted. Should we add the acid in the first tank or in the second? Adding to the second tank is more advisable, because the phase lag of the control loop, when the pH in the outlet of the second tank is measured, is smaller. On the other hand, only acid addition in the first tank makes the system controllable. (We cannot affect the state of the first tank by adding acid in the second tank.)

Attempts have been made to quantify controllability (Lückel and Müller 1975, Müller and Weber 1972, Healey and Mackinnon 1975). It is known that the columns of the matrix $P = (B, AB, \ldots A^{n-1}B)$ span the controllable space. Quantifications are based on how well the space is spanned, i.e., the angle between the columns of P. Implications for the engineering quality of the control are not clear.

Consider the following example which would require some measure of controllability: An exothermic reversible first order reaction with rate constants k_1, and k_{-1} respectively

$$A \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} R$$

is carried out in a CSTR, and the output concentration of R is to be controlled. Should we use the coolant temperature/flowrate or the flowrate of the reactants (residence time) as a manipulated variable?

For the question of controllability, it is of no importance if there is actually a cooling coil in the reactor or if the reactants are precooled. We assume the second case for our model. The mass and energy balance equations are

$$\frac{dA^*}{dt} = \frac{1}{\tau^*}(A_o - A^*) - k_1A^* + k_{-1}R^* \quad (4)$$

$$\frac{dR^*}{dt} = \frac{1}{\tau^*}(R_o - R^*) - k_{-1}R^* + k_1A^* \quad (5)$$

$$\frac{dT^*}{dt} = \mathscr{H}(k_1A^* - k_{-1}R^*) + \frac{1}{\tau^*}(T_0^* - T^*) \quad (6)$$

where $\mathscr{H} = -\Delta H/\rho C_p$, ΔH is the heat of reaction, τ^* the residence time, ρ the density, and C_p the heat capacity. We linearize the equations around the assumed steady state denoted by a subscript s in the following and arrive at

$$\dot{x} = Fx + Gu$$

where

$$x^T = [A, R, T] \text{ and } u^T = [\tau, T_o]$$

with A, R, T, T_0 and τ, denoting and derivations from the steady state, i.e., $A = A^* - A_s^*$ etc. and $r_A = k_1A$, $r_R = k_{-1}R$. Also

$$F = \begin{bmatrix} -\left(\frac{1}{\tau_s} + k_{1s}\right) & k_{-1s} & \frac{\partial}{\partial T}(r_R - r_A)\Big|_s \\ k_{1s} & -\left(\frac{1}{\tau_s} + k_{-1s}\right) & \frac{\partial}{\partial T}(r_A - r_R)\Big|_s \\ \mathscr{H}k_{1s} & -\mathscr{H}k_{-1s} & \mathscr{H}\left[\frac{\partial}{\partial T}(r_A - r_R)\Big|_s - \frac{1}{\tau_s}\right] \end{bmatrix} \quad (7)$$

$$G = \begin{bmatrix} \frac{-A_o + A_s}{\tau_s^2} & 0 \\ \frac{-R_o + R_s}{\tau_s^2} & 0 \\ \frac{-T_{os} + T_s}{\tau_s^2} & \frac{1}{\tau_s} \end{bmatrix} \quad (8)$$

Applying the rank condition for controllability, we conclude that the system will be generally controllable for most steady states, with either τ or T_o as manipulated variables. If the reactor was designed to achieve maximum conversion $[\partial/\partial T(r_R - r_A)_s = 0]$, the reactor is controllable with τ but not with T_o. Consequently, it is very difficult to control a reactor using the cooling rates as a control input, if it was designed to operate in the neighborhood of the maximum conversion point. A quantitative controllability criterion should express this difficulty.

The noncontrollability can be explained well on physical grounds: If we are at the point of maximum conversion, any change in temperature will decrease the concentration of R in the outlet. Measuring R gives us no information on which side of the hill we are, and if more cooling will increase or decrease R. A feedback control becomes impossible.

The dual property of observability guarantees—roughly speaking—that by observing the output

$$y = C(t)x \quad y \in R^r \quad (2)$$

of the system (1) for some time we can reconstruct its state, or expressed differently, by measuring y, no part of the system escapes our observation. If A and C are constant matrices the pair (C, A) is completely observable if and only if

$$\text{rank } [C^T, A^TC^T, \ldots (A^T)^{n-1}C^T] = n \quad (9)$$

Observability does not suffer from similar deficiencies as the ones we mentioned for controllability.

Uncontrollability and unobservability can also be interpreted (Rosenbrock 1970, 1974) as pole—zero cancellation in the transfer function (decoupling zeros). If the system is not observable or not controllable, and if the cancelled pole lies in the right half-plane, we cannot stabilize the system through any feedback arrangement.

Extension of the Basic Concepts

Here, we extend the conditions for controllability and observability to make them suitable for control structure synthesis. Three different procedures will be outlined to arrive at essentially the same result. We believe that the equivalence between the different notions is developed here for the first time, and that it adds significant insight into these new definitions.

For regulation purposes, we want to construct a number of feedback loops which keep the controlled variables at their setpoints, in spite of disturbances entering the system. Assume that we want to achieve the goal using PI control. The system of equations is extended to include integral action

$$\begin{pmatrix} \dot{x} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} A & 0 \\ C & 0 \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} u \quad (10)$$

It is easy to see that $z = \int_{t_0}^{t_1} Cx dt = \int_{t_0}^{t_1} y dt$. When state feedback is used, $u = K\binom{x}{z}$, we will have PI control action on z. Let us explore the consequences of requiring controllability of the pair

$$\left(\begin{pmatrix} A & 0 \\ C & 0 \end{pmatrix}, \begin{pmatrix} B \\ 0 \end{pmatrix}\right)$$

for the feasibility of a control structure. Conditions for controllability of this pair were derived by Porter and Power (1970) and Davison and Smith (1971). A simple proof is given by Morari and Stephanopoulos (1979).

Theorem 1: The pair

$$\left(\begin{pmatrix} A & 0 \\ C & 0 \end{pmatrix}, \begin{pmatrix} B \\ 0 \end{pmatrix}\right)$$

is controllable if and only if the following two conditions are satisfied

1) (A, B) is controllable

$$2) \text{ rank} \begin{pmatrix} A & B \\ C & 0 \end{pmatrix} = n + r, \quad (11)$$

where,

$$r = \dim(y), \text{ and } n = \dim(x).$$

The first condition is quite obvious, but the second deserves some interpretation. For the rank to be equal to $n + r$, rank $(B) \geq r$, we need at least as many manipulated variables as there are measurements. The observation matrix C and the controller matrix B are included in the condition. Consequently, we get information about the feasibility of the commonly applied control structures where the number of inputs matches the number of outputs.

In the context of hierarchical control, Findeisen (1976) arrived at a condition similar to (2) of Theorem 1 whose derivation adds additional insight. Assume the static nonlinear model of the system is given by

$$f_i(x, u) = 0 \quad i = 1, \ldots n \quad (12)$$

and we choose the observations to be represented by

$$c_j(x, u) = y \quad j = 1, \ldots r \quad (13)$$

(The observations are allowed to include a direct dependence on the manipulated variable u, which is slightly more general than what we assumed for the linear case). The observations satisfy the condition that the control action u, necessary to put the system outputs at the desired setpoints y, should be uniquely given through the relations (12) and (13). We can invoke the Implicit Function Theorem to determine if this is possible

$$\text{rank} \begin{bmatrix} \frac{\partial f_i}{\partial x_k} & \frac{\partial f_i}{\partial u_l} \\ \frac{\partial c_j}{\partial x_k} & \frac{\partial c_j}{\partial u_l} \end{bmatrix} = n + r \quad (14)$$

This matrix is the Jacobian of f and c, with respect to x and u and the derivatives are computed at the steady state operating point. If we had linearized the system initially then

$$\frac{\partial f_i}{\partial x_k} = A \quad \frac{\partial f_i}{\partial u_l} = B \quad \frac{\partial c_j}{\partial x_k} = C \quad \frac{\partial c_j}{\partial u_l} = D$$

and if the observations are not directly dependent on the inputs u, $(D = 0)$ (11) and (14) are identical. We can regard condition (11) as a uniqueness condition. It will be shown later that controllability of (A,B), observability of (C,A) and condition (11) are independent and do not imply each other. Condition (14) is not to be confused with a test concerning the multiplicity of steady states. Through the use of (14), we check if a particular steady state can result from several different control actions, while in the usual context of multiplicity, we determine if several different steady states can be caused by the same control action.

Rosenbrock (1970, 1974) introduced the notion of functional controllability. A system is functionally controllable, if given any "suitable" vector $y(t)$ of output functions defined for $t > 0$, there exists a vector $u(t)$ of inputs defined for $t > 0$, which generates the output vector $y(t)$ from the initial condition $x(0) = 0$. ("Suitable" refers to some smoothness conditions and that the variables must be Laplace transformable).

Theorem 2 (Rosenbrock, 1970): A system is functionally controllable if and only if $\dim(u) \geq \dim(y)$ (the inequality brings redundance and so we assume $\dim(u) = \dim(y)$) and

$$\det(G(s)) \not\equiv 0$$

where $G(s)$ is the transfer function matrix relating y and u:

$$G(s) = C(sI - A)^{-1}B$$

Using Schur's formula (Gantmacher 1959)

$$\det \quad G(s) = -\det\begin{pmatrix} sI - A & B \\ C & 0 \end{pmatrix} \Big/ \det(sI - A)$$

and consequently it is sufficient for det $(G(s)) \not\equiv 0$ that

$$\det\begin{pmatrix} -A & B \\ C & 0 \end{pmatrix} \not\equiv 0$$

which is the same as (11) and (14). It might be surprising that we do not need controllability for functional controllability. The answer to this lies in the definition of $x(0) = 0$. The necessary and sufficient conditions for functional controllability guarantee that y will always lie in the controllable subspace. Therefore, starting at the origin, always an element of the controllable subspace, the control action can never lead us into the uncontrollable subspace. Whether the uncontrollable subspace (outside the origin) is empty or not is irrelevant for functional controllability. If we start at a point different from the origin, we can generally not generate a desired output y by choosing appropriate inputs u.

STRUCTURAL CONTROLLABILITY AND OBSERVABILITY

Feedback Control

To check the rank conditions for controllability, observability and the condition (11), quite elaborate models of the system are needed, and due to the linearization the results, will be valid in the immediate surrounding of the point of linearization only. Controllability conditions for nonlinear systems are a topic of current research (Gershwin and Jacobson 1971) and are not easily applicable for our purposes. A compromise would be to require pointwise controllability and observability in a certain region (This does *not* imply controllability of the nonlinear system).

As mentioned earlier, the numerical rank condition for controllability and observability depends on the fortuitous selection of the parameter values, and for well behaved physical systems it fails at isolated points only. Thus, it does not provide any useful, global information about the behavior of a controlled system. It is this last aspect which dictates that any meaningful information should depend on the invariant structural aspects of a dynamic system. This approach clearly has advantages and disadvantages. One advantage is the ease of modelling. A disadvantage is the impossibility of drawing quantitative conclusions as were needed in the reactor example treated above.

Lee and Markus (1967) were the first to point out that uncontrollability is often caused by one special choice of parameters in the matrices A and B, but the structural representation clearly eliminates this possibility. It is not possible to determine the structural controllability of a system by just computing (2) in structural terms and checking its rank. The conditions for structural controllability for single input systems were first derived by Lin (1976), and extended and modified by Shields and Pearson (1976) and Glover and Silverman (1976).

Some basic definitions (Lin 1974, Shields and Pearson 1976) are needed for the developments.

Definition 1: A structured matrix A is a matrix having fixed zeros in certain locations and arbitrary entries (denoted by x) in the remaining locations.

Definition 2: A structured system (A,B) is an ordered pair of structured matrices.

Definition 3: The systems (A,B) and $(\bar{A},\bar{B})$ are structurally equivalent if there is a one-to-one correspondence between the locations of the fixed zeros and nonzero entries of the corresponding matrices of each system.

Definition 4 (structural controllability): The system (A,B) is structurally controllable if, and only if, for every $\epsilon > 0$, all the structurally equivalent systems $(\bar{A},\bar{B})$ are controllable in the usual sense (i.e., satisfy Kalman's numerical rank condition) where $\|A - \bar{A}\| < \epsilon$ and $\|B - \bar{B}\| < \epsilon$.

Definition 4 provides the essence of the structural controllability and relates this new concept to the established notion of Kalman's state controllability. The following theorem can be proved easily (Shields and Pearson (1976) Glover and Silverman (1976) and Morari (1977)).

Theorem 3: The structural pair (A,B) is controllable if, and only if, the following two conditions are satisfied 1) each state node is accessible from at least one control node, and 2) the generic rank of the structural compound matrix $(A;B)$ is n. The first n columns of $(A;B)$ are the columns of A, the last columns those of B.

A graphical interpretation of those two conditions is instructive. A and B are given as

$$A_1 = \begin{pmatrix} x & x & 0 \\ x & x & 0 \\ x & x & x \end{pmatrix} \quad B_1 = \begin{pmatrix} 0 \\ 0 \\ x \end{pmatrix}$$

The graph corresponding to (A_1,B_1) is given in Figure 3. From the control note B, we have access to the state node 3 only. Nodes 1 and 2 are nonaccessible, therefore the pair (A_1,B_1) is not structurally controllable. The physical implication is clear: In the example of the two CSTRs in series above, the concentration in the first tank is not accessible if we decide to add acid in the second tank.

On the other hand, let A and B be given as

$$A_2 = \begin{pmatrix} 0 & x & 0 \\ 0 & x & 0 \\ 0 & x & 0 \end{pmatrix} \quad B_2 = \begin{pmatrix} x \\ x \\ x \end{pmatrix}$$

Figure 4. shows that all the state nodes are accessible, but the structural rank of (A_2,B_2) is two. The graph shows three state nodes, but only two different inputs to those nodes (the outputs of 2 and B). We can interpret the generic rank deficiency as a lack of degrees of freedom.

If B_1 and B_2 are control variables for the DEE, from Figure 2, only nodes 1, 2 and 4 will be accessible. On the other hand C_F, for example, allows access to node 3, and from there, all the other nodes can be reached. To make the compound matrix (A,B) for the double effect evaporator nonsingular, at least two control variables must be chosen ($\rho_g(B) \geq 2$), because the A matrix contains two zero columns. Choosing $B = (T_F, S_i)$ makes $\rho_g(A;B) = 4$, consequently, this choice is not permissible. If C_F and T_F are disturbances which we cannot influence, there are four possible manipulated variables left: F, S_i, B_1, B_2. The following pairs satisfy conditions 1 and 2 of Theorem 3: (F, S_i), (F, B_1), (F, B_2), (S_i, B_1), (S_i, B_2). To satisfy controllability, any control structure should contain one of the pairs above among the manipulated variables.

Observability is known to be dual in the algebraic sense to controllability. This duality expresses itself through the two test criteria (2) and (9). The duality can be interpreted also in the structural sense, by stating that the structural pair (C, A) is observable if, and only if, the structural pair (A^T, C^T) is controllable. The graph of A^T is the graph of A with the directions of the edges reversed, and it is shown for the evaporator in Figure 5. We choose our observations $y = Cx$ to satisfy structural observability. For accessability to nodes 1 and 2, we have to measure W_1 and W_2. Those observations guarantee also that $\rho_g(A^T,C^T) = 5$. We conclude that any control structure should contain measurements of W_1 and W_2.

While the accessibility requirement is quite clear, more insight into the rank condition is gained by looking at the frequency domain. Uncontrollability and unobservability are associated with the existence of decoupling zeroes or pole/zero cancellations. In the case of plant parameter uncertainty, i.e., if each parameter is represented as a variable with a range of values, equivalent to our structural interpretations, Horowitz and Shaked (1975) proved that no pole/zero cancellation can occur at any point of the real axis, except at the origin when all the state nodes are accessible.

In other words, if the accessability conditions are satisfied for structural controllability and observability, no unstable mode can escape our influence and observations, with the exception of

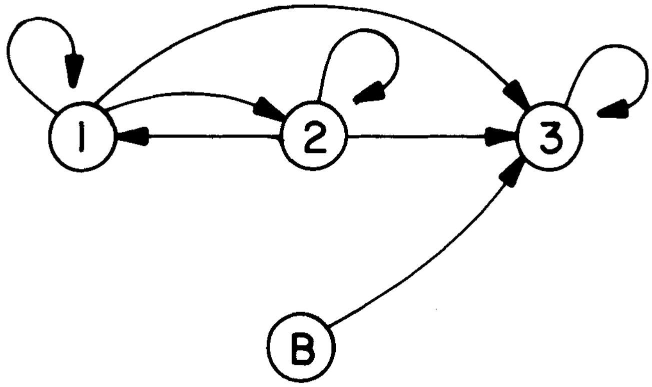


Figure 3. Digraph representation of a system with nonaccessible state nodes (1, 2).

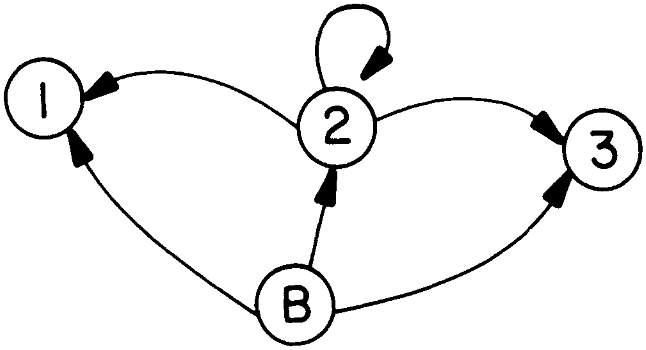


Figure 4. Digraph representation of a system exhibiting generic rank deficiency.

poles at $s = 0$. The same is true for the stable modes, but there is no concern about their existence anyway, as long as they are not very slow. Hautus (1969) gives a different test for the controllability and observability of a plant.

Theorem 4: The pair (A, B) is controllable if, and only if, rank $(A - \lambda I,B) = n$ for all eigenvalues λ of the matrix A. If the rank test fails for λ_i, the mode associated with this eigenvalue is not controllable.

The seconu condition for structural controllability is $\rho_g(A,B) = n$, which is equivalent by the above theorem to testing for the existence of a pole/zero cancellation at the origin ($\lambda = 0$). The generic rank condition serves to detect pure integrators which are not controllable with a given set of manipulated variables. Its

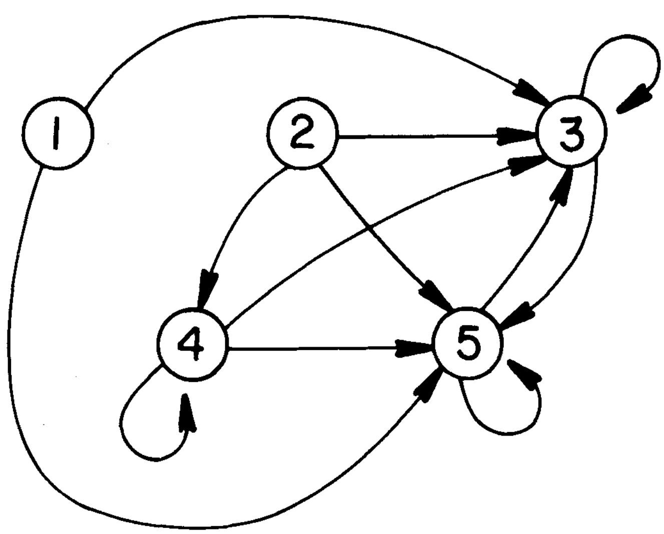


Figure 5. The dual digraph for the double effect evaporator.

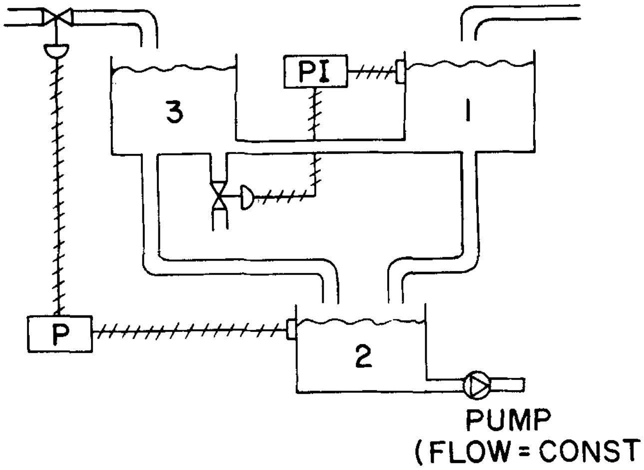


Figure 6. Three interacting watertanks—example of an infeasible control structure.

importance for control structure synthesis purposes is larger than the accessibility condition, because integrators offer a potential for instability and should be subjected to control action. Nonaccessibility to a state node which is not an integrator is irrelevant, as long as we know from physical reasoning that the overall system has no poles in the right half-plane. When the accessibility condition is violated, pole/zero cancellations other than at the origin are present. The same can be said about the two conditions for observability. For a stable system, accessibility to states which are not pure integrators is not crucial. The two-tank pH-control example emphasizes this point: As long as the levels in the two tanks are controlled, the system is stable. Nonaccessibility to the state of the first tank is, therefore, not important. We might just as well add the acid in the second tank, to satisfy the generic rank condition.

In the previous section, we introduced the integral control controllability criterion with its equivalent versions and demonstrated its usefulness for the control structure synthesis procedure. The translation of the conditions into structural terms is straightforward. Thus,

Theorem 5: The structural pair

$$\left[\begin{pmatrix} A & 0 \\ C & 0 \end{pmatrix}, \begin{pmatrix} B \\ 0 \end{pmatrix}\right]$$

is structurally controllable if, and only if, 1) (A, B) is structurally controllable, and 2) ρ_g

$$\begin{pmatrix} A & B \\ C & 0 \end{pmatrix}$$

$= n + r$. For the proof of this theorem, see Morari (1977).

Note that a necessary condition for ρ_g

$$\begin{pmatrix} A & B \\ C & 0 \end{pmatrix}$$

$= n + r$ is that $\rho_g(A,B) = n$ and $\rho_g(A^T,C^T) = n$, i.e., the rank conditions for structural controllability and structural observability have to be satisfied but need not be checked separately. On the contrary, controllability and observability do not imply

$$\rho_g \begin{pmatrix} A & B \\ C & 0 \end{pmatrix} = n + r$$

nor can they be deduced from that condition, as shown with simple examples:

$$A_1 = \begin{pmatrix} x & x & 0 \\ x & x & 0 \\ 0 & 0 & x \end{pmatrix} \quad B_1 = \begin{pmatrix} 0 \\ 0 \\ x \end{pmatrix} \quad C_1 = (0 \quad 0 \quad x)$$

This system is neither controllable nor observable but

$$\rho_g \begin{pmatrix} x & x & 0 & 0 \\ x & x & 0 & 0 \\ 0 & 0 & x & x \\ 0 & 0 & x & 0 \end{pmatrix} = 4$$

Even more important

$$A_2 = \begin{pmatrix} x & 0 & x \\ x & 0 & x \\ x & x & x \end{pmatrix} \quad B_2 = \begin{pmatrix} 0 \\ 0 \\ x \end{pmatrix} \quad C_2 = (x \quad 0 \quad 0)$$

This triple is controllable and observable, but

$$\rho_g \begin{pmatrix} x & 0 & x & 0 \\ x & 0 & x & 0 \\ x & x & x & x \\ x & 0 & 0 & 0 \end{pmatrix} = 3$$

and therefore a feedback structure using manipulated variables and observations corresponding to B_2 and C_2 is infeasible. If we develop control structures on the basis of cause and effect only (Govind and Powers 1976), i.e., just check accessibility, we might be become victim to the illusion that (C_2, A_2, B_2) is a control structure which can be applied in practice. Physical analogs to the structure (C_2,A_2,B_2) are easy to construct. An example in the form of three interacting water tanks is shown in Figure 6. When a pipe connection is drawn to the bottom of a tank, the liquid level in the tank is assumed to influence the flowrate. The outflow of tank 2 is constant, a P controller is already installed, and the feasibility of a PI controller between the level in tank 1 and the drainage valve of tank 3 is to be tested. It is important to point out that simple heuristics, cause-and-effect arguments, etc. are insufficient to draw conclusions about the infeasibility of the proposed control structure in this nontrivial situation. Further, the structural representation allows the immediate conclusion that, if the valve were placed at the interconnection of tank 1 and 3 ($B_2^T = (x \ 0 \ x)$), the control structure is feasible. This result is, again, difficult to recover from any other method.

Feedforward Control

Feedforward structures can also be developed through a structural analysis by introducing the concept of disturbability, somewhat different from the concept as used by Shah *et al.* (1977). Thus,

Definition 5: A state node in the graph of A is said to be disturbable when it is accessible from a disturbance node. As an example, in the double effect evaporator (Figure 2), a disturbance in the feed concentration C_F, affects all state nodes. A disturbance in the outlet of tank 1, B_1, affects W_1, W_2 and C_2 only. The intent of using feedforward control is to eliminate the effect of a particular disturbance on some state or output variable through the action of a manipulated variable.

Conditions for the feasibility of a feedforward control structure can be related to those of feedback structures:

Assume that the influence of the disturbance d on the set of state nodes S is to be eliminated. Select a set of state nodes S^* whose removal renders the set S undisturbable by d (a possible choice is $S^* = S$). Select a set of manipulated variables M, for which S is undisturbable after removal of S^*. If the generic rank condition is satisfied for M and the observations S^*, the feedforward structure with M is feasible.

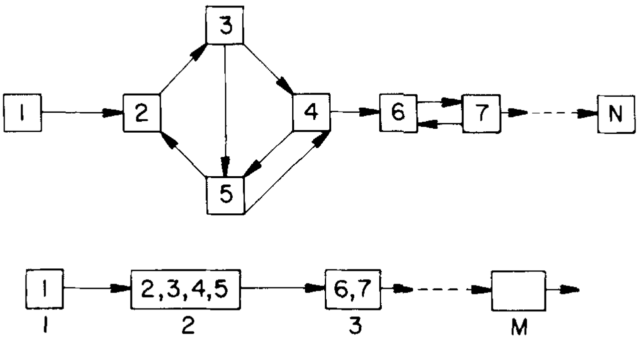


Figure 7. Grouping and precedence ordering of a system.

As an example, for the double-effect evaporator (DEE): $d = F$ = feed flowrate $S = S^* = C_2$

feasible choice of M: B_1
infeasible choice of M: B_2

As a second example, DEE: $d = T_F$ = feed temperature

$S = C_2, W_1, W_2$
$S^* = h_1$
feasible choice of M: S_i
infeasible choice of M: F

DECOMPOSITION FOR THE SYNTHESIS OF REGULATORY STRUCTURES

In Part I, criteria were developed to guide the designer in the decomposition of a large system for optimization. We are left with the problem of designing the regulators 1) to implement the actions prescribed by the optimizing control level and 2) to keep the system operation within the imposed constraints, despite the influence of disturbances. Again, the synthesis problem is very complex. We could easily develop a computer program which checks for an entire plant the feasibility of the different possible sets of manipulated and controlled variables according to the structural rank and accessibility criteria. The number of resulting solutions would be enormous, and screening them would represent an almost insurmountable task. Govind and Powers (1976) use essentially this approach employing heuristics to screen, in addition to known control-theoretical considerations like speed of response, time lags, etc.

Instead of eliminating the majority of the alternatives after they have been synthesized, it appears preferable not to generate them at all. This can be achieved by decomposing the process, synthesizing the regulators only within the subsystem's boundaries, and finally combining the subsystems appropriately. Guiding principles for the decomposition must be established.

Decomposition has to be performed along arguments involving system dynamics. With increasing "distance" between measurement and control action, the time lag and the phase angle of the open loop system increase. Tight control becomes impossible, the resonant frequency of the closed loop response to disturbances decreases, and with it, the disturbances which pass through the system largely undamped increase. It is not reasonable to generate control structures which suggest manipulating a variable at the first stage of a process in order to influence a variable in the last stage. This can be easily avoided if decomposition precedes the synthesis, and if the control objectives of a subsystem are met—as much as possible—through the manipulation of variables located in the same subsystem. The following operations are applied to the integrated plant:

1. Precedence order and group the units (e.g. Sargent and Westerberg, 1964). Through that algorithm, we obtain a chain of groups in sequential order (see Figure 7). The groups are irreducible, i.e., they contain recycle connections and therefore the units forming a group cannot be arranged in a sequence.

2. Determine the minimum number of "torn streams" in the irreducible groups (Barkley and Motard 1972, Pho and Lapidus 1973) and break up the irreducible groups into chains of functional units e.g., reactor *with* associated heat exchanger, distillation column *with* condenser, reboiler, preheater and pumps, etc.

3. Generate the control structures for each of the sequentially arranged functional units spearately along the following principles:

a. Number the units sequentially in the direction of flow (1, . . N). (For the following discussion we assume for simplicity the absence of branches)

b. Define for each unit i the set of control objectives S_i' and the variables which can be manipulated M_i'.

c. For the last functional unit $i = N$ form

$$S_N = S_N' \quad \text{(set of measurements)}$$

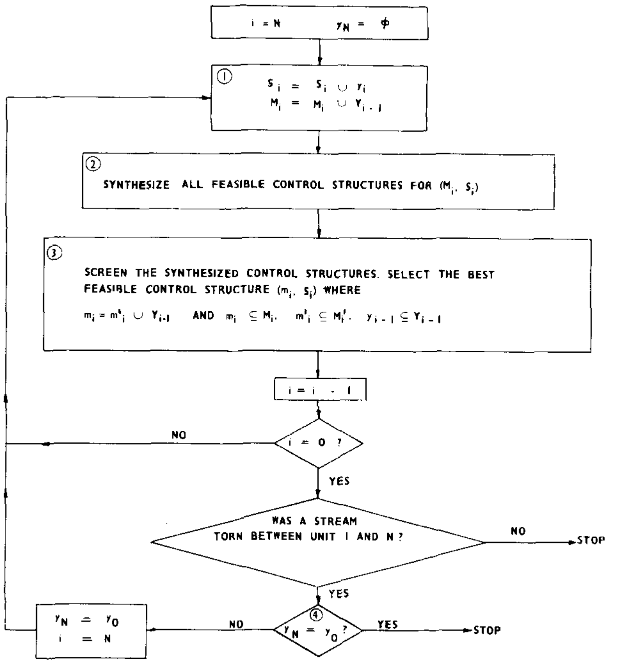


Figure 8. Flow chart for the synthesis of control structures employing decomposition.

and

$$M_N = M_N' \cup Y_{N-1} \quad \text{(set of manipulated variables)}$$

Synthesize all the feasible control structures for (M_N, S_N). Then, screen the synthesized control structures and select the best feasible control structure (m_N, S_N) where;

$$m_N = m_N' \cup y_{N-1},\ m_N \subseteq M_N,\ m_N' \subseteq M_N' \text{ and } y_{N-1} \subseteq Y_{N-1}$$

d. Proceed with functional unit $i = N - 1$ and repeat the same synthesis procedure as in (c) above. Continue with the rest until $i = 0$. Thus for the general unit i we have

$$S_i = S_i' \cup y_i \text{ and } M_i = M_i' \cup Y_{i-1}$$

e. Synthesize all the feasible control structures for (M_i, S_i) and screen them to identify the best (m_i, S_i), where

$$m_i = m_i' \cup y_{i-1},\ m_i \subseteq M_i,\ m_i' \subseteq M_i' \text{ and } y_{i-1} \subseteq Y_{i-1}$$

The algorithmic procedure described in (c) and (d) above, is also presented in the flowchart of Figure 8. Certain remarks are in order to clarify the synthesis strategy.

(1) The set of inputs Y_{i-1} from the preceding unit can formally be regarded as an augmentation of the directly manipulatable variables. Thus the complete set of manipulated variables available to unit i is $M_i = M_i' \cup Y_{i-1}$. If a subset $y_{i-1} \subseteq y_{i-1}$ was indeed chosen as "manipulated variables" in unit i, then y_{i-1} has to become a control objective for unit $i - 1$, i.e., $S_{i-1} = S'_{i-1} \cup y_{i-1}$.

(2) The procedure for the synthesis of all feasible control structures for every unit i follows the algorithm described in the next section.

(3) The screening has to be carried out on the basis of steady state considerations first, like gains, and then through the use of dynamic models of increasing complexity. As a heuristic, $m_i \in M_i'$ should generally be preferred to $m_i \in Y_{i-1}$ ("local control"). If $S_i \neq S_i'$, the screening has to include the unit $(i + 1)$ as well, and might lead to a change in the set m_{i+1}. It is possible that several almost equivalent control structures are synthesized for a unit. Then several structures can be synthesized for the preceding unit and so on.

(4) If an irreducible group was broken up and $y_o \neq \phi$, then the control structure assumes some of the torn variables to be manipulated. Thus y_N has to be modified and a iterative procedure is initiated.

THE ALGORITHM FOR GENERATING FEASIBLE ALTERNATIVE CONTROL STRUCTURES

At this point, all the primary regulatory control objectives (quality and level of production, safety, environmental and other regulations), and all the secondary regulatory control objectives arising from the development of feedback optimizing control structures have been defined. Synthesizing the alternative regulatory schemes will be done at the level of the units resulting from the decomposition. For each unit i proceed to synthesize the feasible control structures starting with

Step 0: Translate the regulatory control objectives into the set of process variables P_i; e.g. if S_i is the product quality, then P_i is the purity concentration of the desired chemical in the product stream.

Step 1: Choose as observations the process variables P_i (or functions of them) which correspond to the defined primary or secondary regulatory control objectives S_i. If some of them are not measurable, select secondary measurements according to the theory developed in Part III. Let the set of the selected observations be represented by the matrices C_i and D_i; i.e. $y_i = C_ix_i + D_iu_i$.

Step 2: Test for the dual accessibility (Appendix A) using the observations selected in step 1. Augment the set of observations if necessary to achieve accessibility. Call the augmented observation vector $y_i^* = C_i^*x_i + D_i^*u_i$.

Step 3: Form the structural matrix

$$\begin{pmatrix} A_i & B_i \\ C_i^* & D_i^* \end{pmatrix}$$

where

$$\begin{pmatrix} B_i \\ D_i^* \end{pmatrix}$$

are the columns formed by all feasible manipulated variables M_i of subsystem i; i.e. $\dot{x}_i = A_ix_i + B_iu_i$.

Step 4: Delete columns from

$$\begin{pmatrix} B_i \\ D_i^* \end{pmatrix}$$

so that the number of remaining manipulated variables, corresponding to the remaining

$$\begin{pmatrix} \bar{B}_i \\ \bar{D}_i^* \end{pmatrix},$$

is equal to the number of observations for subsystem i.

Step 5: Test for accessibility (Appendix A) in the system (A_i, $\bar{B}_i$). If accessibility is not satisfied the set of manipulated variables selected is not feasible and it is rejected. If accessibility is satisfied the set of manipulated variables selected is retained for further screening (step 6).

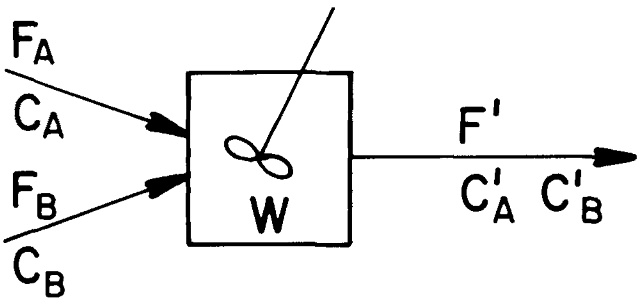


Figure 9. Mixer-blender (example 1).

Step 6: Test if

$$\begin{pmatrix} A_i & \bar{B}_i \\ C_i^* & \bar{D}_i^* \end{pmatrix}$$

is structurally nonsingular, i.e., if it does not exhibit generic rank deficiency (Appendix B). If the test is affirmative (i.e., system is structurally nonsingular) the selected observations which correspond to the rows of the matrix C_i^*, and manipulated variables, and which correspond to the columns of

$$\begin{pmatrix} \bar{B}_i \\ \bar{D}_i^* \end{pmatrix},$$

represent a feasible regulatory structure. Otherwise, reject the set of selected manipulated variables as infeasible.

Some remarks: The accessibility tests in steps 2 and 5 can be relaxed if the open-loop system is known to have no poles in the open right half-plane. In that case, test for nonaccessibility of integrators only, where an integrator arises when a variable has a fixed zero diagonal element.

Almost every chemical plant is connectable, in the sense of Davison (1977). Therefore it is not necessary that every isolated subsystem is accessible using the manipulated variables and observations associated with it. To generate all the alternative structures it is advisable to eliminate columns from B_i one at a time and to construct an "elimination tree". An illustration is given in the example section.

For quick hand calculations, the FZRF algorithm (Appendix B) is recommended to determine if a matrix has full rank. Depending on the available method for computing determinants Davison's (1977) method might be more practical for computer calculations.

EXAMPLES

The following examples exhibit an increasing degree of difficulty. Of course, for simple processes the resulting control strategy will be the same as that developed by intuitive methods, but the practice gained on simple systems is invaluable when complex systems are encountered. As is true for all general synthesis strategies (e.g. synthesis of heat exchangers, separation sequences, etc.) the merit of the synthesis method is to generate initially all feasible structures, some of which might have been overlooked otherwise. This allows the generation of completely new and hopefully attractive, alternatives.

Example 1: Mixer-Blender Without Heat Effects

The system is shown in Figure 9. The model is general, can be used for any stream junction or stream split as part of a general plant. The development of the structural matrix is straightforward

	W	C'_A	C'_B	F_A	F_B	F'	C_A	C_B
W	0	0	0	x	x	x	0	0
C'_A	0	x	0	x	x	0	x	0
C'_B	0	0	x	x	x	0	0	x

Let the objective be the control of C'_A. Choosing C'_A as the only observation, there is dual nonaccessibility to the nodes W and C'_B starting from the node C'_A. Observation of W is crucial (it is an integrator leading to instability), so we have to extend the set of observations to (W, C'_A). Next, the structural matrix is augmented to include these observations

				6	1	2	3	4	5
		W	C'_A	C'_B	F_A	F_B	F'	C_A	C_B
1	W	0	0	0	x	x	x	0	0
2	C'_A	0	x	0	x	x	0	x	0
3	C'_B	0	0	x	x	x	0	0	x
		x	0	0			0		
		0	x	0					

It is trivial that the first two columns of the structural matrix shown above are structurally independent and can be eliminated before performing the generic rank test. The indices (1, 2, . . . , 6) shown above indicate the re-ordering of the columns. The reordered structural matrix with the blocks R_i (see Appendix B), marked by rectangles, is

	F_A	F_B	F'	C_A	C_B	C'_B
W	x	x	x	0	0	0
C'_A	x	x	0	x	0	0
C'_B	x	x	0	0	x	x

This reordering of the structural matrix facilitates tremendously the generic rank test of the composite matrix. According to the developments in Appendix B, for the present mixer-blender example we construct the following matrices $\bar{R}_i$

$$\bar{R}_1 = \overset{\begin{matrix} F_A & F_B & F' \end{matrix}}{[\, x \quad x \quad x \,]} \qquad \bar{R}_2 = \overset{\begin{matrix} F_A & F_B & F' & C_A \end{matrix}}{\begin{pmatrix} x & x & x & 0 \\ x & x & 0 & x \end{pmatrix}}$$

and

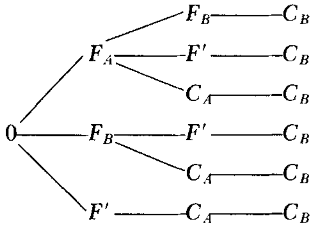
$$\bar{R}_3 = \overset{\begin{matrix} F_A & F_B & F' & C_A & C_B & C'_B \end{matrix}}{\begin{pmatrix} x & x & x & 0 & 0 & 0 \\ x & x & 0 & x & 0 & 0 \\ x & x & 0 & 0 & x & x \end{pmatrix}}$$

The rank indices (RI) for the above three matrices are

$$\text{for } \bar{R}_1, \quad (RI)_1 = 2$$
$$\text{for } \bar{R}_2, \quad (RI)_2 = 2$$
$$\text{for } \bar{R}_3, \quad (RI)_3 = 3$$

As explained in Appendix B, the rank index $(RI)_i$ of the matrix $\bar{R}_i$ indicates the number of columns exceeding the number of rows in the matrix $\bar{R}_i$. Consequently, $(RI)_i$ is the maximum number of columns of $\bar{R}_i$ which can be deleted without affecting the generic rank of the overall composite matrix for the dynamic system. In the case of the mixer-blender example, we can delete at most the columns F_A and F_B or F_A and F' or F_B and F' without affecting the generic rank, since $(RI)_1 = 2$. Similarly, we can delete three columns from $\bar{R}_3$ (at most, three variables from the set F_A, F_B, F', C_A, C_B, C'_B). However, not all the three deleted columns may be columns of $\bar{R}_1$ for which we have $(RI)_1 = 2$. These simple properties for the structural submatrices $\bar{R}_i$ allow us to construct graphically the "elimination tree to present alternative choices of manipulated variables.

From the matrix $\bar{R}_3$, we notice that we can select F_A, or F_B, or F' as the first manipulated variable to be eliminated. As discussed above in terms of the rank indices, for each choice F_A, F_B or F' there are two more manipulated variables eligible for elimination. Thus, the nodes F_A, F_B, F' define the three primary branches of the elimination tree. Further branches define the additional manipulated variables to be eliminated. For example; after eliminating F_B, the eligible variables for elimination, are F' and C_A. Similarly, for F_A the eligible variables are F_B, F' or C_A; and for F' it is C_A. The following elimination tree is developed



At the last stage of the tree development, the choice is unique, since there are at most three variables which can be eliminated. This elimination tree is equivalent to the following six sets of alternative manipulated variables

$$(F', C_A) \quad (F_B, C_A) \quad (F_B, F') \quad (F_A, C_A) \quad (F_A, F') \quad (F_A, F_B)$$

For a detailed discussion of the above algorithmic procedure, refer to Appendix B.

If the blender is merely a pipe junction we certainly do not need feedback control to control the level. In this case the procedure eliminates automatically one degree of freedom (one manipulated variable) and overspecification becomes impossible.

Example 2: Double Effect Evaporator

Selecting C_2 as our objective and observation we find W_1 and W_2 are nonaccessible. The augmented set of observations is (W_1, W_2, C_2). These reordered structural matrix is

	C_1	h_1	B_1	B_2	F	C_F	T_F	S_i
C_2	x	x	x	0	0	0	0	0
W_2	x	x	x	x	0	0	0	0
W_1	x	x	x	0	x	0	0	0
C_1	x	x	0	0	x	x	0	0
h_1	x	x	0	0	x	0	x	x

The rank indices for the consecutive matrices $\bar{R}_i$ $i = 1,2,3,4,5$ are 2,2,2,2 and 3 respectively. One of the eliminated variables has to be T_F or S_i. Two others can be chosen from the set B_1, B_2, F, C_F. One possible structure could consist of the manipulated variables F, S_i and B_2 and the measurements W_1, W_2 and C_2. If single loops are employed a likely pairing would be W_1-S_i, W_2-B_2 and C_2-F. If, for example, the pump regulating the outlet of tank 1, B_1, is limiting the capacity under certain conditions, this appears to be a reasonable choice.

Example 3: Distillation Column

This example illustrates the procedure on a staged system. The structural relationships for a (pseudo-) binary distillation process are derived from mass- and energy balance considerations.

Around plate i

$$\frac{dx_i}{dt} = f_1(V_{i-1}, x_{i-1}, V_i, x_i, L_{i+1}, x_{i+1}, P; L_F, x_F) \tag{15}$$

$$\frac{dH_i}{dt} = f_2(V_{i-1}, L_i, V_i, L_{i+1}; L_F) \tag{16}$$

$$\frac{dL_i}{dt} = f_3(H_i, L_i, L_{i+1}; L_F) \tag{17}$$

$$\frac{dV_i}{dt} = f_4(H_i, V_{i-1}, V_i; V_F) \tag{18}$$

For the condenser

$$\frac{dH_c}{dt} = g_1(V_n, L_c) \tag{19}$$

$$\frac{dx_c}{dt} = g_2(x_c, x_n, P, V_n) \tag{20}$$

$$\frac{dP}{dt} = g_3(V_F, V_B, V_n) \tag{21}$$

For the reboiler

$$\frac{dH_B}{dt} = h_1(V_B, L_1, L_B) \tag{22}$$

$$\frac{dx_B}{dt} = h_2(V_B, L_1, x_B, x_1, P) \tag{23}$$

Notation:

x = mole fraction of (pseudo) component in liquid
V = molar vapor flux leaving plate/reboiler
L = molar liquid flux leaving plate/reboiler/condenser
H = molar holdup
P = pressure

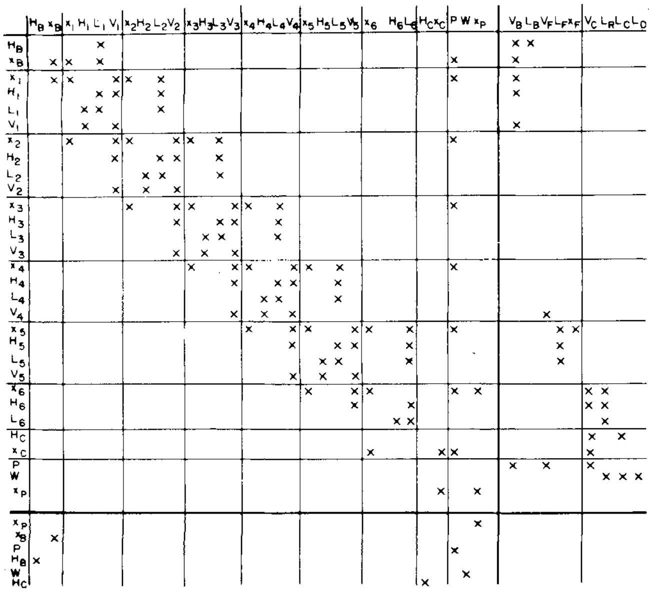


Figure 10. Structural matrix of the distillation column (example 3).

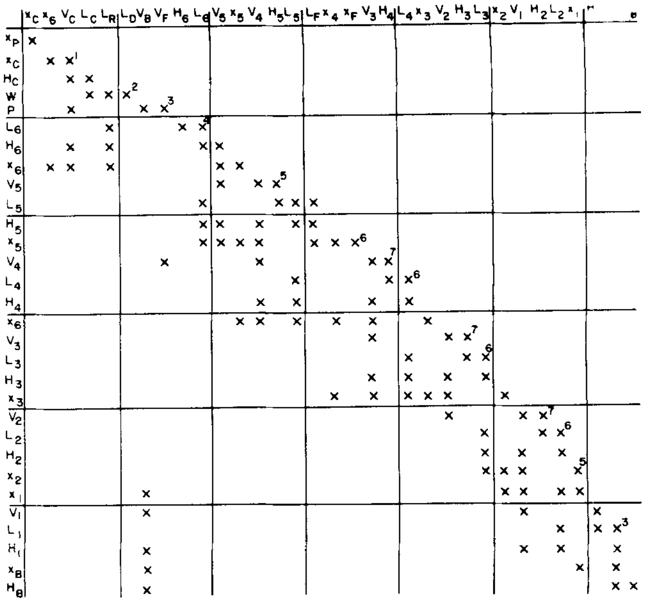


Figure 11. Reordered structural matrix of the distillation column (example 3).

W = reflux drum holdup

Subscripts:

i = plate number (counting from bottom to top, $i = 1, \ldots n$)
B = bottom
C = condenser
F = feed
R = reflux
P = product (distillate)

Some remarks: Equation (17) approximates the hydrodynamic relationship on the tray i. At steady state with the assumption of equal molal overflow, $L_i = L_{i+1}$. The transient behavior will depend on the holdup. Equation (18) can be regarded as a structural approximation to the enthalpy balance the assumptions being negligible enthalpies of the liquid streams, equal molal overflow in the steady state and dependence of the transient behavior on the holdup. On the top plate, V_n is given by the condenser duty. The change in column pressure (21) is modeled as a balance of the vapor entering the column (V_F), the vapor generated by the reboiler (V_B) and the condensed vapor V_n.

The structural matrix is presented in Figure 10. At the bottom, the equations for the reflux drum are added. Note that a minimal number of trays was used according to Appendix C.

We choose as our objectives and observations, the bottom composition x_B, the distillate composition x_D, and the column pressure P. We can easily find by inspection or through the proposed algorithm that H_B, W and H_c have to be measured also for observability purposes. These columns can be eliminated from our scheme, and rows and columns reordered as shown in Figure 11. Points of changing indices for the different blocks R_i are marked by the corresponding index. The last index (3) is equal to the number of manipulated variables which may be deleted. The indices show that we may eliminate a maximum of

two variables from the set
V_c, L_c, L_R, L_D

three variables from the set
$V_c, L_c, L_R, L_D, V_B, V_F, L_F, x_F, L_B$

The eliminated variables act as disturbances, or can be used for optimizing control, the remaining ones as manipulated variables. At least one of the variables classified originally as "manipulated" will be production limiting. Thus it will be at its extreme setting, and it cannot be used for regulation purposes. We should try to eliminate those manipulated variables which are impractical (e.g., x_F) and which might reach their saturation value during operation. Assume for example that the steam for the feed preheater is supplied by a different unit (not subject to our manipulation), and that x_F is regarded as a disturbance. Again the "elimination tree" can be developed.

The number of possibilities is restricted because x_F and V_F are deleted for all control structures. After minor reordering of the matrix, any one of the other manipulated variables can be deleted, with the rest forming a feasible control structure. We could for example, use the boilup rate as an optimization variable or require the distillate flowrate to be fixed. The remaining variables have to be paired with the measurements appropriately. A possible slightly unorthodox combination for the latter situation is shown in Figure 12. Note that we assume the actual outlet compositions to be measured for control, which is often not done in practice. The choice of secondary measurements (e.g. temperatures) to estimate the outlet composition is a different question, treated in Part III.

Example 4: Williams-Otto Plant

A model of a medium size chemical plant (Williams and Otto 1960) serves as a test case for process computer applications.

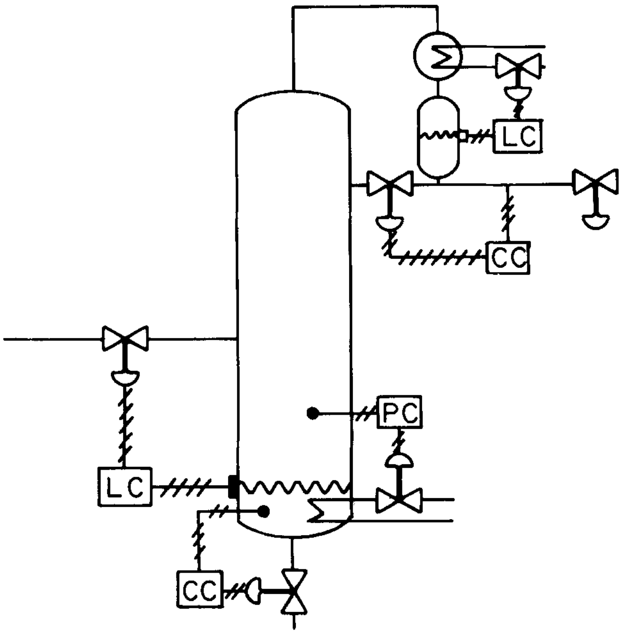


Figure 12. A feasible control structure for the distillation column (example 3).

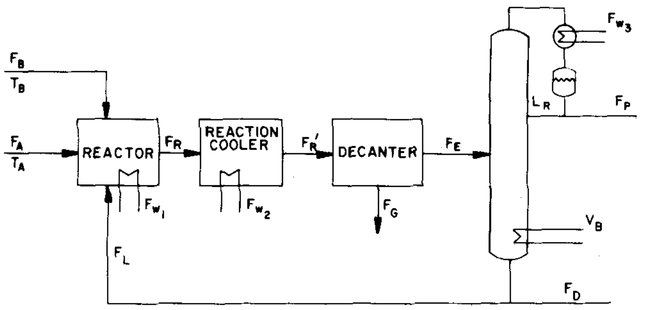


Figure 13. The Williams-Otto plant.

Because of its general nature (many standard unit operations and a recycle stream) it was used frequently in the past for optimization and control studies, see Dibella and Stevens (1965), Adelman and Stevens (1972), Luus and Jaakola (1973), Davison and Chadha (1972), Jung *et al.* (1971). The reactions

$$A + B \rightarrow C$$

$$C + B \rightarrow P + E$$

$$P + C \rightarrow G$$

are carried out in a cooled stirred tank reactor to produce the desired product P. C and E have fuel value and G is an insoluble waste material (tar), separated in a decanter after the reaction mixture has been cooled. The product is separated from the byproducts and reactants through distillation, part of which are recycled to the reactor. The main features of the plant and the interconnections in Figure 13.

The model is essentially the same as described by Williams and Otto (1960), with some minor modifications. 1) The holdup in the reactor and the decanter were included as variables. 2) The first order lag in the reaction cooler was omitted because it has no influence on the feasibility arguments. 3) The distillation column was modeled as in the last example with the assumption of no vapor feed. Closer inspection shows that one tray less is necessary in the structural model than used in the previous example.

The objectives defined here were slightly different from the ones assumed before. 1) Keep the reactor variables at levels determined by the optimization layer of the control structure. 2) The reaction cooler should cool the product mixture down to 100°F(38°C), otherwise G cannot be separated in the decanter. 3) The bottom temperature in the distillation column should not exceed a maximal value to avoid reaction among the components. 4) The top product concentration should meet the specifications.

Under the assumption that the reactant concentrations in the recycle stream are approximately constant, objective 1) can be expressed as 1a) Keep the reactor temperature at the specified level. 1b) Keep the inlet stream ratios F_A/F_L and F_B/F_L constant. 1c) Keep the residence time constant by adjusting the active volume of the reactor according to the total flowrate (If it is not varying significantly, this is equivalent to keeping the level in the reactor constant).

Notation:

$A, B, C,$
E, G, P = concentrations of corresponding products in Reactor (subscript R) and in decanter (subscript E)
F_A, F_B = inlet flows of components A, B respectively
T_A, T_B = inlet temperatures of components A, B
F_D = distillation column, unrecycled bottom flowrate
F_E = distillation column—total feed flowrate
F_G = "tar" flowrate out of decanter
F_P = distillation column, product flowrate
F_R = total flowrate out of reactor
F_{W_1}, F_{W_2}, F_{W_3} = cooling water flowrate in reactor, reaction cooler and distillation column condenser respectively
L_C = distillation column, condensate flowrate out of condenser
L_R = distillation column reflux flow rate
T_B = distillation column, bottom temperature
T_R = temperature in reactor
T_R' = temperature after reaction cooler
V_B = distillation column, steam rate into reboiler
V_E = total holdup in decanter
V_E' = "tar" holdup in decanter
V_R = holdup in reactor
x_i = concentration of component i

The notation for the distillation column is described in Example 3, with the exceptions noted above.

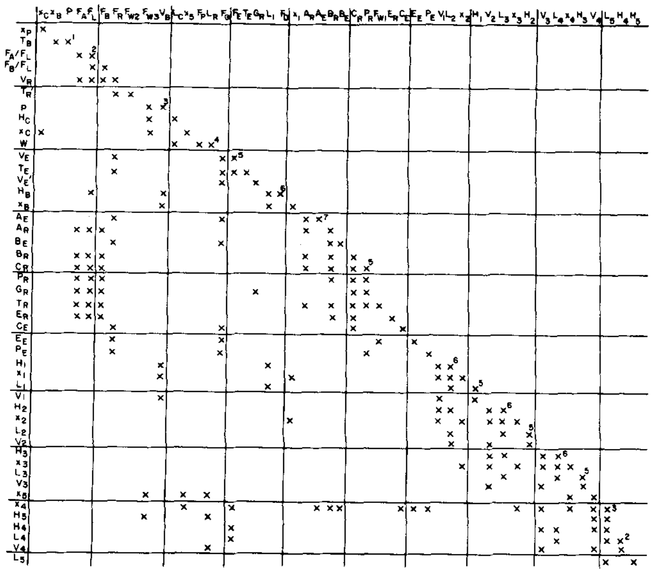


Figure 14. Reordered structural matrix of the Williams-Otto plant.

For illustrative purposes, we solved the problem once with and once without decomposition. For the solution without decomposition, the structural matrix for the entire system was developed. In accordance with the stated objectives, the following were chosen as controlled variables: F_A/F_L, F_B/F_L, T_R, T_R', x_p, V_R and $T_B = f(x_B,P)$. To achieve structural observability, this set of observations had to be augmented by V_B, V_E, V_E', H_C, W, H_B. After rearrangement, the structural matrix shown in Figure 14 is obtained. The rules in Appendix B show that a total of two variables may be eliminated from the available manipulated variables; the remaining are to be used in feedback loops. We could develop all the feasible sets of manipulated variables but will center on a specific problem instead: If F_A is subject to conditions in a different part of the plant, it cannot be used as a manipulated variable, and the corresponding column can be eliminated from the structural matrix. Another manipulated variable can be freely eliminated, as long as the remaining matrix satisfies the structural rank condition. We found that the following alternatives are available

$$F_{W_3},\ V_B,\ L_C,\ F_P,\ L_R,\ F_D,\ F_{W_1}$$

If the condenser of the distillation column is limiting, i.e. F_{W_3} had to be eliminated, a control structure of the type shown in Figure 15 results.

If we put other constraints on the system (e.g., required product rate, condenser limitings) the control structure will have to be different. The structural matrix (Fig. 14) (after eliminating columns corresponding to the constraints) will allow immediate deduction of the feasible sets of manipulated variables, if any exist.

The chief disadvantage of this integrated approach is the dimensionality of the problem: At the end, large sets of manipu-

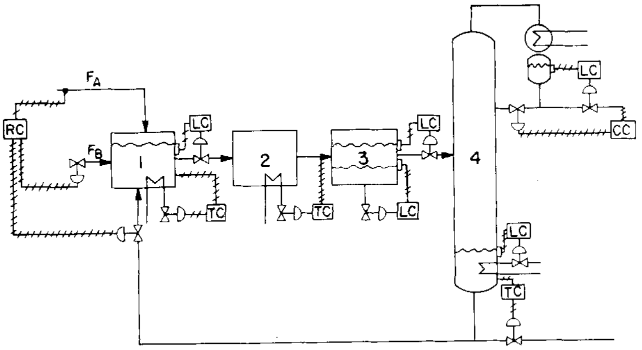


Figure 15. One possible control structure synthesized for the Williams-Otto plant (no decomposition).

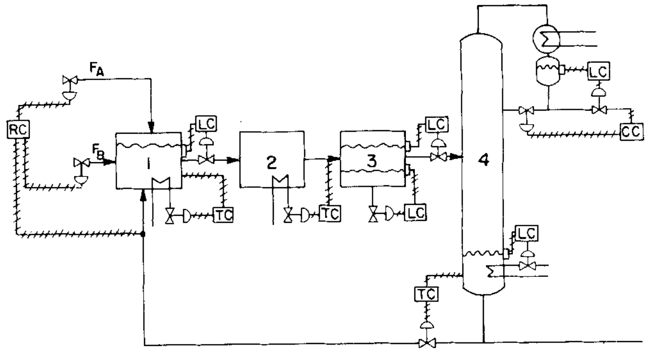


Figure 16. Alternate control structure synthesized for the Williams-Otto plant (with decomposition).

lated and controlled variables have to be paired, which is a nontrivial task. This problem is largely alleviated through the use of decomposition.

For the solution with decomposition, we use the notation of section 4. The units are numbered as in Figure 15. The whole plant represents one irreducible group of units. The recycle stream is torn. Then, according to the logic of Figure 8, we have for Unit 4:

$$S_4' = S_4 = (T_B, x_p)$$
$$M_4 = M_4' \cup Y_3$$
$$= (V_B, F_L, F_D, F_P, L_R, L_C, F_{W3}) \cup F_E$$

We can proceed exactly as in Example 4. The structural matrix is a guide to augmenting S_4 to satisfy the accessibility conditions. Then the different feasible sets of manipulated variables are developed and we choose, say,

$$m_4 = (V_B, F_L, F_p, L_R, L_C)$$
$$y_3 = \phi$$

in agreement with the stated heuristic that it is preferable to have $y_i = \phi$ to achieve faster system response. For Unit 3:

$$S_3' = S_3 = \phi$$
$$M_3 = (F_E, F_G) \cup F_R'$$

Two observations are required for accessibility and we choose

$$m_3 = (F_E, F_G)$$
$$y_2 = \phi$$

Unit 2:

$$S_2' = S_2 = T_R'$$
$$M_2 = (F_R, F_R', F_{W2})$$

One additional observation is required for accessibility and we choose

$$m_2 = (F_R', F_{W2})$$
$$y_1 = \phi$$

Unit 1:

$$S_1' = S_1 = (F_A/F_L, F_B/F_L, V_R, T_R)$$
$$M_1 = (F_A, F_B, F_{W1}, F_R) \cup F_L$$

From the structural matrix it can be deduced that no additional observations are necessary. We choose

$$m_1 = (F_A, F_B, F_{W1}, F_R)$$
$$y_4 = \phi$$

When synthesizing the control structure for Unit 4, we assumed $y_4 = \phi$. We find now at the beginning of this chain of units that indeed $y_4 = \phi$. Thus there is no incompatibility across the torn stream, and the control structure is indeed feasible for the entire plant. A possible pairing of the selected controlled and manipulated variables is shown in Figure 16. Assume now that F_A cannot be used as a manipulated variable. Then we have to choose

$$m_1 = (F_B, F_{W1}, F_R) \cup F_L$$
$$y_4 = F_L$$

Thus, the original assumption $y_4 = \phi$ was incorrect. We have to return to unit 4 and revise our assumption (recycle loop in flow diagram in section 4)

$$S_4 = (T_b, x_p) \cup F_L$$

A set of manipulated variables achieving objectives S_4 and satisfying the rank condition would be

$$m_4 = (V_B, F_L, F_D, F_P, L_R, L_C)$$

The sets of manipulated and controlled variables are now identical with those obtained by the approach without decomposition (Figure 15).

ACKNOWLEDGMENT

Financial support from the National Science Foundation through the Grant ENG 75-11165 is gratefully acknowledged.

NOTATION

The nomenclature of the variables related to the various examples is not included in the following table. It is given in the main text of the paper, at the appropriate places.

- A = plant matrix of a finite-dimensional dynamic system
- B = input matrix of a finite-dimensional dynamic system
- C = output matrix of a finite-dimensional dynamic system; output matrix for the state variables
- c = process output variables regulated at a given level
- D = output matrix for the controlled variables
- d = vector of external disturbances
- f = vector of transformation equations for nonlinear steady state systems
- G = plant transfer function matrix
- I = identity matrix
- M = set of all manipulated variables available for a system including interconnection streams available for manipulation
- M' = set of manipulated variables associated with a given system or subsystem
- m = vector of manipulated variables in an active control structure
- R = reachability matrix for a system of interconnected units.
- S = set of all measured variables available to a system (or subsystem) including additional augmentation for integrated systems
- S' = set of measured variables associated with a given system or subsystem
- $\bar{S}$ = structural matrix of complete feedback configurations
- s = variable of the Laplace transform

t = time
u = vector of input variables
x = vector of state variables
Y = set of all output variables associated with a given system
y = vector of output (measured) variables
z = vector of "artificial" state variables denoting integral feedback action

Greek Letters

λ = eigenvalues of the plant matrix A
ρ_g = generic rank of a structural matrix

Subscript

i = ith subsystem or unit

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APPENDIX A

Accessibility

Results for testing accessibility in a digraph are readily available. Associated with every digraph is the adjacency matrix, defined as follows: The adjacency matrix A of a digraph with n nodes and no parallel edges is an nxn Boolean matrix whose element

$x_{ij} = 1$ if there is an edge directed from the i^{th} node to the j^{th} node

$= 0$ otherwise.

The adjacency matrix is the transpose of the structural system matrix defined before with all the x entries replaced by 1.

The reachability matrix R is defined to be a function of A

$$R = (I + A + \ldots A^{n-1}) = (I + A)^{n-1}$$

and it can be shown (Roberts 1976) to have the following property: Node j is reachable from node i if, and only if, $r_{ij} = 1$.

The following algorithm was developed to find R, and it appears simpler because it avoids taking the power of a Boolean matrix.

1) $P^{(1)} = I + A$

2) $P^{(i+1)} = P^{(i)} + Q^{(i)} \qquad i = 2, \ldots n - 1$

3) $P^{(n-1)} = R$

The rows of $Q^{(i)}$ are given by

$$q_{j\cdot}^{(i)} = \sum_{k=1}^{n} p_{jk}^{(i)} \times p_{k\cdot}^{(i)}$$

where $q_{j\cdot}^{(i)}$ is the j^{th} row of $Q^{(i)}$, and $p_{k\cdot}^{(i)}$ is the k^{th} row of $P^{(i)}$. Any multiplication and addition in the expressions above has to be performed according to the rules of Boolean algebra. Another simple test was developed by Glover and Silverman (1976). However, it does not allow for as much flexibility in choosing different B's without repeating all the calculations.

An easy test for checking accessibility for a given set of manipulated variables is the following: A set of manipulated variables having nonzero entries in the rows i of the B matrix yields no nonaccessible state nodes if the Boolean sum of the rows i of the matrix R has no zero elements.

APPENDIX B

The Generic Rank of a Structural Matrix

Davison (1977) suggests a simple method for finding the generic rank of a matrix. Stating that $\rho_g(M) = n$ is equivalent to requiring that rank $(M) = n$ for almost all choices of free elements in M. We can choose arbitrary values for the free elements (e.g., using a pseudo-random number generator) and compute the rank of M. To develop confidence in the solution, we can repeat the calculation with a different set of free parameters. The procedure is numerical, and does not guarantee that we find the correct generic rank.

Shields and Pearson (1976) developed a different algorithm, called fixed-zero-rank-finder (FZRF). It can be used to test for controllability directly without distinguishing between condition 1 and 2 of Theorem 3, but we saw that accessibility is not required if the system is stable. And, the test might restrict our choice of control structures in an unnecessary fashion. It can be shown through counter examples that the algorithm as given in the reference is incorrect (Morari and Stephanopoulos 1978).

A different method was adopted to generate the alternative sets of manipulated variables which satisfy the necessary condition of Theorem 5 for a given set of observations.

$$\text{Let } \bar{S} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where the submatrices are defined by Equation (14). Alternative structures can exist only if the number of columns of S exceeds the number of rows. We would like to find those columns of

$$\begin{pmatrix} B \\ D \end{pmatrix}$$

which when deleted do not reduce $\rho_g(S)$. The remaining columns of

$$\begin{pmatrix} B \\ D \end{pmatrix}$$

form a feasible set of manipulated variables for feedback control using the observations C.

In the context of the solution of large systems of equations, Steward (1965) developed an algorithm to find an output set assignment. The method is complete in the sense that if no output set can be found, the matrix is structurally singular. No modifications are necessary when the algorithm is used on a nonsquare system. The rows for which no output variables are found are the generically dependent ones. The task of finding an output set is greatly simplified and the use of the "Steward path" is avoided in most cases, when the following ordering algorithm is used initially.

Step 1: $i = 0, \quad j = 0$
Step 2: Find the row of $\bar{S}$ with the least number of entries (say k).
Step 3: $i \rightarrow i + 1$; associate this row with the index i and delete it.
Step 4: Associate the columns which have entries in this row with the indices $j + 1, j + 2, \ldots j + k$ and delete them; $j \rightarrow j + k$
Step 5: If there are rows left go to Step 2, otherwise the end of the ordering algorithm is reached.

Rewrite matrix $\bar{S}$ with the rows having increasing indices from top to bottom and with the columns having increasing indices from left to right. The matrix $\bar{S}$ will now be of the form

$$\begin{bmatrix} R_1 & & & & 0 \\ x & R_2 & & & 0 \\ x & x & R_3 & & 0 \\ x & x & \cdot & \cdot & \\ x & \cdot & \cdot & R_w & \end{bmatrix}$$

where the blocks R_i contain no fixed zero elements and are of dimension $n_i \times m_i$.

Definition: The matrix $\bar{R}_i$ for $i = 1, 2, \ldots, w$ is defined as follows:

$$\bar{R}_i = \begin{bmatrix} R_1 & & & \\ x & R_2 & & \\ x & x & R_3 & \\ \cdot & & & \\ \cdot & & & \\ x & x \ldots & \ldots & R_i \end{bmatrix}$$

Definition: The rank index $(RI)_i$ for matrix $\bar{R}_i$, $i = 1, 2, \ldots, w$ is defined as;

$$(RI)_i = \sum_{k=1}^{i} (m_k - n_k)$$

Remarks:

(1) If $\rho_g(\bar{S}) < n$, $n - \rho_g(\bar{S})$ rows are structurally dependent, i.e., the system contains redundant equations or noncontrollable integrators.

(2) S cannot contain any zero rows or columns otherwise there is a pole/zero cancellation at the origin and different observations and/or manipulated variables have to be chosen.

(3) $(RI)_i$ indicates the number of columns exceeding the number of rows in the matrix $\bar{R}_i$. Consequently $(RI)_i$ denotes the largest number of columns of $\bar{R}_i$ which can be deleted without affecting $\rho_g(\bar{S})$. For example, suppose that $(RI)_2 = 2$ and $(RI)_3 = 4$. Then we can delete at most four columns (which corresponds to possible manipulated variables) of $\bar{R}_3$ without affecting $\rho_g(\bar{S})$. However, at most, two of those columns (manipulated variables) may be columns of $\bar{R}_2$ since $(RI)_2 = 2$.

The algorithm for determining different sets of manipulated variables yielding structural matrices of full generic rank:

1) Find an output set for $\bar{S}$.

2) All manipulated variables which do not appear in the output set can be eliminated. The remaining matrix will be of full generic rank.

(3) A manipulated variable which appears in the current output set can be eliminated if, after deletion of the corresponding column, a different output set can be found using a "Steward path".

APPENDIX C

Structural Representation of Staged Systems

Staged systems are very common in chemical engineering, but the structural matrix of a distillation column with 40 trays, for example, would be very inconvenient to handle. A reduction in size which does not influence the conclusions on the generic rank and the accessibility is called for. The structural matrix of a staged system is generally of the form

$$M = \begin{bmatrix} A' & B' & & & & & & & F \\ C' & A & B & & & & & & 0 \\ & C & A & B & & 0 & & & 0 \\ & & C & A & B & & & & \\ & & & & \cdot & & & & \\ & & & & & \cdot & & & \\ & & & & & & \cdot & & \\ & & 0 & & & C & A & B & \\ & & & & & & C & A & B^* & 0 \\ & & & & & & & C^* & A^* & G \end{bmatrix}$$

where A, B, C, A', B', C', A^*, B^*, C^* are structured matrices.

Theorem on the representation of staged systems: The matrix M is of full rank if the matrix N

$$N = \begin{bmatrix} A' & B' & & & & & F \\ C' & A & B & & & & 0 \\ & C & A & B & & & 0 \\ & & C & A & B & & 0 \\ & & & C & A & B & 0 \\ & & & & C & A & B^* & 0 \\ & & & & & C^* & A^* & G \end{bmatrix}$$

is of full rank, where A', A and A^* are submatrices with n', n and n^* rows respectively and the other occurring matrices are dimensioned accordingly.

Proof: M is of full rank if and only if the removal of any m rows decreases the column rank by exactly m. (It can never decrease by more than m. If it decreases by less, then some of the rows were linearly dependent).

Remove the n' first rows from M; the column rank has to decrease by n', otherwise N could not be of full rank. Remove rows $n' + 1$ to $n'' + n$ from M; again the column rank has to decrease by n otherwise N could not be of full rank. We can continue this procedure removing the next n rows and comparing with N and so on.

Some remarks: The form of M arises naturally. A', B', C' and A^*, B^*, C^* account for the fact that the first and thhe last stage are generally modelled differently from the other stages. F and G represent some control action at the beginning and the end of the staged system.

If measurements are taken at an intermediate stage they should be represented as measurements at the initial or final stage, otherwise the order of the model increases but the results stay the same.

The minimal order is given by the requirement that there has to be one "standard row" A, B and C. Closer inspection can sometimes yield an even smaller representation if not all the different matrices A', B', C', A^*, B^*, C^* are present. This is demonstrated by the example of the distillation column and the reduced model of the column included in the Williams-Otto plant.

Manuscript received February 6, 1979; revision received August 24, and accepted August 27, 1979.